Physically adequate proper reference system of a test observer and relativistic description of the GAIA attitude

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Abstract

A relativistic definition of the physically adequate proper reference system of a test observer is suggested within the framework of the PPN formalism. According to the nomenclature accepted within the GAIA project this reference system is called Center-of-Mass Reference System (CoMRS). The interrelation between the suggested definition of the CoMRS and the Resolutions 2000 on relativity of the International Astronomical Union (IAU) are elucidated. The tetrad representation of the CoMRS at its origin is also explicated. It is demonstrated how to use that tetrad representation to calculate the relation between the observed direction of a light ray and the corresponding coordinate direction in the Barycentric Celestial Reference System of the IAU. It is argued that the kinematically non-rotating CoMRS is the natural choice of the reference system where the attitude of the observer (e.g. of the GAIA satellite) should be modeled. The relativistic equations of rotational motion of a satellite relative to its CoMRS are briefly discussed. A simple algorithm for the attitude description of the satellite is proposed.

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I. INTRODUCTION

Future space astrometry projects like GAIA [1–3] and SIM [4] are expected to attain an accuracy of 1 microarcsecond (μas) for positions of remote celestial sources. This high accuracy requires general relativity to be used for data modeling. A relativistic model of positional observation with microarcsecond accuracy involves many subtle details. Recently a number of such models have been suggested (see [5–8] and references therein). The purpose of this paper is, first of all, to provide a relativistic definition of the physically adequate local (proper) reference system of a test observer (e.g., a satellite). According to the nomenclature adopted within the GAIA project [9] this reference system is called Center-of-Mass Reference System (CoMRS) below. As a co-product, the calculation of the observed light direction as adopted in [7] is explained in detail and explicitly justified. The CoMRS is intended to be used to describe physical phenomena located within the immediate vicinity of the observer (e.g., the rotational motion of the satellite, the process of observation, etc.). This reference system can be used to define the attitude parameters of the satellite. In order to define the CoMRS we make use of the parametrized post-Newtonian (PPN) version of the relativistic framework adopted recently by the International Astronomical Union (IAU) for the use for high-precision astrometry, celestial mechanics, geodesy and metrology [10]. The IAU Resolution B1.3 (2000) adopted at the XXIV General Assembly [10–12] of the IAU specifies a global reference system, the Barycentric Celestial Reference System (BCRS), and a physically adequate local geocentric reference system, the Geocentric Celestial Reference System (GCRS), in the framework of the post-Newtonian approximation of general relativity. Below it is argued that a simplified version of the GCRS constructed for a massless observer as a central body can be used as a physically adequate CoMRS.

The problem of constructing a physically adequate proper reference system of a massive body (e.g. Earth) in the first post-Newtonian approximation has been thoroughly discussed by several authors. In the framework of general relativity two advanced formalisms are available. One formalism is due to Brumberg and Kopeikin [13–17] and another one is due to Damour, Soffel and Xu [18–21]. For the gravitational N-body problem both formalisms introduce a total of $N+1$ different coordinate systems: one set of global coordinates (t, x^i) and one set of local comoving coordinates (T, X^a) for each body. Note that in this context a body is just a material subsystem at the boundaries of which the energy-momentum tensor vanishes. In the local coordinates the metric tensor possesses the following two properties:

A. The gravitational field of external bodies is represented only in the form of a relativistic tidal potential which is at least of second order in the local spatial coordinates and coincides with the usual Newtonian tidal potential in the Newtonian limit.

B. The internal gravitational field of the central body coincides with the gravitational field of a corresponding isolated source provided that the tidal influence of the external matter is neglected.

These two requirements can be simultaneously satisfied in general relativity as has been shown in the framework of the Brumberg-Kopeikin and Damour-Soffel-Xu formalisms. It is clear that this fact is closely related to the validity of the Strong Equivalence Principle in general relativity. These two formalisms complement each other by elaborating the theory from slightly different points of view. The formalisms deliver (1) an elegant description of metric tensors in both the global and local coordinates and the closed-form transformations

between them, (2) an improved description of the structure of the gravitational field of each body by means of a set of its multipole moments which are linked in an operational way to what can be observed in the local gravitational environment of the body, (3) a description of the influence of the external gravitational fields in the local reference system by means of some suitably defined tidal moments, (4) post-Newtonian translational and rotational equations of motion of the N bodies with full multipole structure, (5) physically adequate equations of motion of a test particle in a local reference system, (6) physically adequate relativistic models for many kinds of observations (VLBI, high-accuracy positional observations, remote clock synchronization, etc.). The IAU 2000 Resolution B1.3 [10] that defines the metric tensors of both the global BCRS and the local GCRS, is based on the results on these two approaches.

In the framework of the PPN formalism with two parameters β and γ the theory of physically adequate local reference systems was developed in [22, 23]. It is clear that because of possible violation of the Strong Equivalence Principle in some alternative theories of gravity it is, generally speaking, impossible to construct a local reference system possessing both properties **A** and **B**. In [23] it has ben shown how to construct local reference systems which possess either property **A** or property **B**. It has been also demonstrated that for relativistic modeling of astronomical observations one should normally prefer the local reference system with property **A**. The theory of local PPN reference systems with the PPN parameters has been developed in [22, 23] with the same degree of details as it was done in general relativity. For the limit of general relativity $\beta = \gamma = 1$ the formulas of [23] coincide with those of the Brumberg-Kopeikin and Damour-Soffel-Xu formalisms as well as with the formulas from the IAU 2000 Resolutions.

The problem of defining physically adequate local coordinates for a massless body (test observer) is much more simple than the problem for a massive body (e.g. for Earth). This problem has been discussed many times in the literature. Let us mention, for example, the work of Ni and Zimmermann [24] where a local reference system of an accelerated observer has been constructed explicitly up to the terms of third order relative to the local spatial coordinates. As it has been noted in [17], the results of the Brumberg-Kopeikin and Damour-Soffel-Xu formalisms can be directly applied to define in an elegant way a physically adequate local reference system of a massless body. Indeed, one should just consider the limit where the gravitational potential of the central body vanishes. It is clear that the same procedure can be applied also to the local PPN reference system [23]. Exactly this will be done below. The resulting reference system represents a natural choice for the physically adequate CoMRS of a massless observer. This reference system can be applied to model physical phenomena in the immediate vicinity of the observer. Two examples will be given below: the relation between the observed direction toward a light source and the relevant BCRS parameters of the light ray, and the rotational motion of the test observer (satellite).

Let us summarize the most important notations used throughout the paper:

- G is the Newtonian constant of gravitation;
- c is the velocity of light;
- β and γ are the parameters of the parametrized post-Newtonian (PPN) formalism which characterize possible deviation of the physical reality from general relativity theory ($\beta = \gamma = 1$ in general relativity);
- the lower case latin indices i, j, k, ... take values 1, 2, 3;

- the lower case Greek indices α, β, ... take values 0, 1, 2, 3;
- repeated indices imply the Einsteinian summation irrespective of their positions (e.g. $a^i\,b^i = a^1\,b^1 + a^2\,b^2 + a^3\,b^3$, $a^\alpha\,b_\alpha = a^0\,b_0 + a^1\,b_1 + a^2\,b_2 + a^3\,b_3$);
- parentheses surrounding a group of indices denote symmetrization, e.g., $A_{i(jk)} = \frac{1}{2}\left(A_{ijk} + A_{ikj}\right)$;
- brackets surrounding two indices denote antisymmetrization, e.g., $A_{i[jk]} = \frac{1}{2}\left(A_{ijk} - A_{ikj}\right)$;
- a comma before an index designates the partial derivative with respect to the corresponding coordinates: $A_{,\mu} = \partial A/\partial x^\mu$, $A_{,i} = \partial A(t,\boldsymbol{x})/\partial x^i$; for partial derivatives with respect to the coordinate times t and $\mathcal{T}$ we use special notations $A_{,t} = \partial A(t,\boldsymbol{x})/\partial t$ and $A_{,\mathcal{T}} = \partial A/\partial \mathcal{T}$;
- a dot over any quantity designates the total derivative with respect to the coordinate time of the corresponding reference system: e.g. $\dot{A} = \dfrac{dA}{dt}$.

Sections II, III and IV are devoted to the definitions of the metric tensors of the BCRS and the CoMRS, and the transformations between these two reference system, respectively. The tetrad induced by the CoMRS coordinates at the CoMRS origin is discussed in Section V. Section VI elucidates the equivalence of several possible ways to calculate the observable direction toward a light source from the relevant coordinate quantities defined in the global BCRS. The post-Newtonian equations of rotational motion of a satellite relative to the CoMRS are briefly discussed in Section VII. In Section VIII it is argued that the kinematically non-rotating CoMRS represents a natural choice of a reference system where the attitude of the observer (e.g. of the GAIA satellite) is modeled. A summary of the main results are given in Section IX.

II. THE PPN METRIC TENSOR IN THE BCRS

Let us consider an isolated system of N gravitating bodies. It is clear that the space-time is asymptotically flat and can be covered with a single global coordinate system $x^\mu = (ct, x^i)$ where

$$\lim_{\substack{|\boldsymbol{x}|\to\infty \\ t=\mathrm{const}}} g_{\mu\nu} = \eta_{\mu\nu}, \tag{1}$$

$g_{\mu\nu}$ being the metric tensor in the global coordinate system. Here $\eta_{\mu\nu} = \mathrm{diag}(-1,+1,+1,+1)$ is the Minkowski metric tensor. In the framework of the PPN formalism [25] with two parameters β and γ the metric tensor $g_{\mu\nu}$ in the global reference system can be written as [22, 23]

$$\begin{aligned}
g_{00} &= -1 + \frac{2}{c^2}\,w(t,\boldsymbol{x}) - \frac{2}{c^4}\,\beta\,w^2(t,\boldsymbol{x}) + \mathcal{O}(c^{-5}),\\
g_{0i} &= -\frac{2\,(1+\gamma)}{c^3}\,w^i(t,\boldsymbol{x}) + \mathcal{O}(c^{-5}),\\
g_{ij} &= \delta_{ij}\left(1 + \frac{2}{c^2}\,\gamma\,w(t,\boldsymbol{x})\right) + \mathcal{O}(c^{-4}),
\end{aligned} \tag{2}$$

where δ_{ij} is the Kronecker symbol. Here and below 3-dimensional coordinate quantities ("3-vectors") referred to the spatial axes of the corresponding reference system are set in boldface: e.g. $\boldsymbol{x} = x^i$. A harmonic-like gauge for the global PPN metric tensor is adopted here. Precisely speaking, the global metric tensor satisfies the usual harmonic gauge ($g = \det(g_{\mu\nu})$)

$$\frac{\partial}{\partial x^\alpha}\left((-g)^{1/2} g^{\alpha\beta}\right) = 0 \tag{3}$$

in case of general relativity $\beta = \gamma = 1$. This requires

$$w_{,t} + w^i_{,i} = \mathcal{O}(c^{-2}). \tag{4}$$

In accordance with the standard PPN framework as described, e.g., in [25] the metric potentials w and w^i are assumed to obey the equations

$$w_{,ii} - \frac{1}{c^2}\, w_{,tt} = -4\,\pi\, G\,\sigma + \mathcal{O}(c^{-4}), \tag{5}$$

$$w^i_{,jj} = -4\,\pi\, G\,\sigma^i + \mathcal{O}(c^{-2}), \tag{6}$$

where

$$\sigma = \frac{1}{c^2}\left(T^{00} + \gamma\, T^{kk} + \frac{1}{c^2}\, T^{00}\,(3\gamma - 2\beta - 1)\, w\right) + \mathcal{O}(c^{-4}), \tag{7}$$

$$\sigma^i = \frac{1}{c}\, T^{0i} + \mathcal{O}(c^{-2}). \tag{8}$$

Here, $T^{\mu\nu}$ are the components of the energy-momentum tensor in the global reference system and w in (7) is needed only to Newtonian order where it coincides with the Newtonian potential. Because of requirement (1) the solution of (5)–(6) can be written in the form

$$w^\mu(t,\boldsymbol{x}) = G \int \frac{\sigma^\mu(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|} d^3x' + \frac{1}{2c^2}\, G\, \frac{\partial^2}{\partial t^2} \int \sigma^\mu(t,\boldsymbol{x}')|\boldsymbol{x}-\boldsymbol{x}'| d^3x' + \mathcal{O}(c^{-4}), \tag{9}$$

where $w^0 = w$ and $\sigma^0 = \sigma$. It is clear that this formulas for w and w^i together with the Newtonian continuity equation

$$\sigma_{,t} + \sigma^i_{,i} = \mathcal{O}(c^{-2}) \tag{10}$$

are compatible with the gauge condition (4). The metric (2)–(9) is equivalent to the PPN metric with coordinates $(t_{\rm pN}, x^i_{\rm pN})$ in the standard post-Newtonian gauge as used, e.g., in [25] up to a trivial gauge transformation

$$\begin{aligned} t_{\rm pN} &= t - \frac{1}{c^4}\, \chi_{,t} + \mathcal{O}(c^{-5}), \\ x^i_{\rm pN} &= x^i, \end{aligned} \tag{11}$$

where χ is the superpotential

$$\chi = \frac{1}{2}\, G \int \sigma(t,\boldsymbol{x}')\, |\boldsymbol{x}-\boldsymbol{x}'|\, d^3x' + \mathcal{O}(c^{-2}), \tag{12}$$

so that

$$\chi_{,ii} = w + \mathcal{O}(c^{-2}). \tag{13}$$

For $\beta = \gamma = 1$ the definition (2) and (7)–(9) coincides with the definition of the BCRS [10, 11] given within general relativity.

III. THE PPN METRIC TENSOR IN THE COMRS

The CoMRS is a physically adequate reference system of an observer $(\mathcal{T}, \mathcal{X}^a)$ the mass of which is so small that its influence on the background space-time can be neglected. The metric tensor in the CoMRS can be derived from the metric tensor of the GCRS by setting the gravitational potential of the central body to zero. Below we modify in this way the PPN version of the GCRS as constructed in [23]. Again by setting $\gamma = \beta = 1$ in the formulas below one can restore the formulas which could be derived directly from the GCRS adopted by the IAU [10–12]. The metric tensor in the CoMRS reads

$$\begin{aligned}
\mathcal{G}_{00} &= -1 + \frac{2}{c^2}\,\mathcal{W}(\mathcal{T}, \boldsymbol{\mathcal{X}}) - \frac{2}{c^4}\,\beta\,\mathcal{W}^2(\mathcal{T}, \boldsymbol{\mathcal{X}}) + \mathcal{O}(c^{-5}),\\
\mathcal{G}_{0a} &= -\frac{2(1+\gamma)}{c^3}\,\mathcal{W}^a(\mathcal{T}, \boldsymbol{\mathcal{X}}) + \mathcal{O}(c^{-5}),\\
\mathcal{G}_{ab} &= \delta_{ab}\left(1 + \frac{2}{c^2}\,\gamma\,\mathcal{W}(\mathcal{T}, \boldsymbol{\mathcal{X}})\right) + \mathcal{O}(c^{-4}),
\end{aligned} \tag{14}$$

where

$$\mathcal{W}(\mathcal{T}, \boldsymbol{\mathcal{X}}) = \mathcal{Q}_a(\mathcal{T})\,\mathcal{X}^a + \mathcal{W}_{\mathrm{T}}(\mathcal{T}, \boldsymbol{\mathcal{X}}), \tag{15}$$

$$\mathcal{W}^a(\mathcal{T}, \boldsymbol{\mathcal{X}}) = \frac{1}{2}\,\varepsilon_{abc}\,\mathcal{C}_b(\mathcal{T})\,\mathcal{X}^c + \mathcal{W}^a_{\mathrm{T}}(\mathcal{T}, \boldsymbol{\mathcal{X}}), \tag{16}$$

where $\varepsilon_{ijk} = (i-j)(j-k)(k-i)/2$ is the fully antisymmetric Levi-Civita symbol. Potentials $\mathcal{W}_{\mathrm{T}}(\mathcal{T}, \boldsymbol{\mathcal{X}})$ and $\mathcal{W}^a_{\mathrm{T}}(\mathcal{T}, \boldsymbol{\mathcal{X}})$ are the external tidal gravitational potentials (both are $\mathcal{O}(|\boldsymbol{\mathcal{X}}|^2)$) which describe the manifestation of the external gravitational field in the CoMRS. The terms $\mathcal{Q}_a(\mathcal{T})\,\mathcal{X}^a$ and $\frac{1}{2}\,\varepsilon_{abc}\,\mathcal{C}_b(\mathcal{T})\,\mathcal{X}^c$, $\mathcal{Q}_a$ and $\mathcal{C}_a$ being arbitrary functions of time $\mathcal{T}$, are linear relative to $|\boldsymbol{\mathcal{X}}|$ and describe the translational and rotation motion of the CoMRS. The $\mathcal{Q}_a$ is the acceleration of the momentarily co-moving locally inertial reference system relative to the CoMRS origin. In other words, an accelerometer placed at the CoMRS origin measures $-\mathcal{Q}_a$ (see, Section VIII of [23] where the equations of motion of a test particle relative to the local PPN coordinates were derived). If the observer (satellite) is a drag-free satellite one can set $\mathcal{Q}_a = 0$. If the observer (satellite) is equipped with some kind of thrusters, $\mathcal{Q}_a(\mathcal{T})$ can be used to describe their influence. Non-gravitational forces can be also described by choosing some special model for $\mathcal{Q}_a(\mathcal{T})$. In the following we consider $\mathcal{Q}_a$ as arbitrary function. The $\mathcal{C}_a$ defines the rotational motion of the spatial axes of the CoMRS relative to the momentarily co-moving Fermi-Walker transported locally inertial reference system. Clearly, the equations of test particles relative to the CoMRS with $\mathcal{C}_a \neq 0$ contain Coriolis forces. Possible choices of C_a and its relation to the rotational matrix $\mathcal{R}^a{}_i$ in the

transformation between the CoMRS spatial coordinates and the spatial coordinates of the BCRS will be discussed below.

Here the harmonic gauge conditions are again assumed to be valid in the $\beta = \gamma = 1$ limit. This implies an equation similar to (4) for potentials $\mathcal{W}$ and $\mathcal{W}^a$ that in turn gives

$$\dot{Q}_a\,\mathcal{X}^a + \mathcal{W}_{\mathrm{T},\mathcal{T}} + \mathcal{W}^a_{\mathrm{T},a} = \mathcal{O}(c^{-2}). \tag{17}$$

Now from the results of [22, 23] one gets

$$\begin{aligned}
\mathcal{W}_{\mathrm{T}}(\mathcal{T},\boldsymbol{\mathcal{X}}) = \quad & w(t,\boldsymbol{x}) - w(\boldsymbol{x}_{\mathrm{o}}) - w_{,j}(\boldsymbol{x}_{\mathrm{o}})\, r^j_{\mathrm{o}} \\
& + \frac{1}{c^2}\Big(-2(1+\gamma)\, v^i_{\mathrm{o}}\, \left(w^i(t,\boldsymbol{x}) - w^i(\boldsymbol{x}_{\mathrm{o}}) - w^i_{,j}(\boldsymbol{x}_{\mathrm{o}})\, r^j_{\mathrm{o}} \right) \\
& \qquad + (1+\gamma)\, v^2_{\mathrm{o}}\, \left(w(t,\boldsymbol{x}) - w(\boldsymbol{x}_{\mathrm{o}}) - w_{,j}(\boldsymbol{x}_{\mathrm{o}})\, r^j_{\mathrm{o}} \right) \\
& \qquad + (1+\gamma)\, \dot{w}^{i,j}(\boldsymbol{x}_{\mathrm{o}})\, r^i_{\mathrm{o}}\, r^j_{\mathrm{o}} + \frac{1}{2}\,\gamma\, \ddot{w}(\boldsymbol{x}_{\mathrm{o}})\, r^2_{\mathrm{o}} + \left(\frac{1}{2} - \beta - \gamma \right)\, (a^i_{\mathrm{o}} r^i_{\mathrm{o}})^2 \\
& \qquad + (1 - 2\beta - 2\gamma)\, \mathcal{Q}_a\, \mathcal{R}^a_i\, r^i_{\mathrm{o}}\, a^j_{\mathrm{o}}\, r^j_{\mathrm{o}} - \gamma\, v^i_{\mathrm{o}}\, r^i_{\mathrm{o}}\, \dot{w}_{,j}(\boldsymbol{x}_{\mathrm{o}})\, r^j_{\mathrm{o}} \\
& \qquad + \frac{1}{2}\gamma\, r^2_{\mathrm{o}}\, \mathcal{Q}_a\, \mathcal{R}^a{}_i\, a^i_{\mathrm{o}} + \frac{1}{10}\,(\gamma - 2)\, r^2_{\mathrm{o}}\, \ddot{a}^i_{\mathrm{o}}\, r^i_{\mathrm{o}} \\
& \qquad + 2(1-\beta)\left[w(\boldsymbol{x}_{\mathrm{o}}) + a^i_{\mathrm{o}}\, r^i_{\mathrm{o}} \right]\, \left(w(t,\boldsymbol{x}) - w(\boldsymbol{x}_{\mathrm{o}}) - w_{,j}(\boldsymbol{x}_{\mathrm{o}}) r^j_{\mathrm{o}} \right) \Big) \\
& + \mathcal{O}(c^{-4}),
\end{aligned} \tag{18}$$

$$\begin{aligned}
\mathcal{W}^a_{\mathrm{T}}(\mathcal{T},\boldsymbol{\mathcal{X}}) = \quad & \mathcal{R}^a{}_i \Big\{ w^i(t,\boldsymbol{x}) - w^i(\boldsymbol{x}_{\mathrm{o}}) - w^i_{,j}(\boldsymbol{x}_{\mathrm{o}})\, r^j_{\mathrm{o}} - v^i_{\mathrm{o}}\, \left(w(t,\boldsymbol{x}) - w(\boldsymbol{x}_{\mathrm{o}}) - w_{,j}(\boldsymbol{x}_{\mathrm{o}}) r^j_{\mathrm{o}} \right) \\
& \quad + \frac{2\gamma + 1}{5\,(1+\gamma)}\, r^i_{\mathrm{o}}\, \dot{a}^j_{\mathrm{o}}\, r^j_{\mathrm{o}} - \frac{3\gamma - 1}{10\,(1+\gamma)}\, \dot{a}^i_{\mathrm{o}}\, \boldsymbol{r}^2_{\mathrm{o}} \Big\} + \mathcal{O}(c^{-2}).
\end{aligned} \tag{19}$$

Here $r^i_{\mathrm{o}} = x^i - x^i_{\mathrm{o}}(t)$, $x^i_{\mathrm{o}}(t)$ are the coordinates of the origin of the local reference system relative to the global one, and $v^i_{\mathrm{o}} = dx^i_{\mathrm{o}}/dt$ and $a^i_{\mathrm{o}} = d^2x^i_{\mathrm{o}}/dt^2$ are its velocity and acceleration, respectively. For any function of $A(t,\boldsymbol{x})$ we use the shorthand notation $A(\boldsymbol{x}_{\mathrm{o}}) = A(t,\boldsymbol{x}_{\mathrm{o}}(t))$.

IV. TRANSFORMATION FROM THE BCRS TO THE COMRS

The coordinate transformations between the BCRS and the CoMRS read [22, 23]

$$\mathcal{T} = t - \frac{1}{c^2}\left(\mathcal{A} + v^i_{\mathrm{o}}\, r^i_{\mathrm{o}} \right) + \frac{1}{c^4}\left(\mathcal{B} + \mathcal{B}^i\, r^i_{\mathrm{o}} + \mathcal{B}^{ij}\, r^i_{\mathrm{o}}\, r^j_{\mathrm{o}} + \mathcal{C}(t,\boldsymbol{x}) \right) + \mathcal{O}(c^{-5}), \tag{20}$$

$$\mathcal{X}^a = \mathcal{R}^a_j \left(r^j_{\mathrm{o}} + \frac{1}{c^2}\left(\frac{1}{2} v^j_{\mathrm{o}}\, v^k_{\mathrm{o}}\, r^k_{\mathrm{o}} + \mathcal{D}^{jk}\, r^k_{\mathrm{o}} + \mathcal{D}^{jkl}\, r^k_{\mathrm{o}}\, r^l_{\mathrm{o}} \right) \right) + \mathcal{O}(c^{-4}), \tag{21}$$

$$\dot{\mathcal{A}}(t) = \frac{1}{2} v^2_{\mathrm{o}} + w(\boldsymbol{x}_{\mathrm{o}}), \tag{22}$$

$$\dot{\mathcal{B}}(t) = -\frac{1}{8}\, v_{\rm o}^4 + 2(\gamma+1)\, v_{\rm o}^i\, w^i(\boldsymbol{x}_{\rm o}) - \left(\gamma + \frac{1}{2}\right)\, v_{\rm o}^2\, w(\boldsymbol{x}_{\rm o}) + \left(\beta - \frac{1}{2}\right)\, w^2(\boldsymbol{x}_{\rm o}), \tag{23}$$

$$\mathcal{B}^i(t) = -\frac{1}{2}\, v_{\rm o}^2\, v_{\rm o}^i + 2(1+\gamma)\, w^i(t, \boldsymbol{x}_{\rm o}) - (2\gamma+1)\, v_{\rm o}^i\, w(\boldsymbol{x}_{\rm o}), \tag{24}$$

$$\mathcal{B}^{ij}(t) = -v_{\rm o}^{(i}\, \mathcal{R}^a_{j)}\, \mathcal{Q}^a + (1+\gamma) w^{(i,j)}(\boldsymbol{x}_{\rm o}) - \gamma\, v_{\rm o}^{(i}\, w^{,j)}(\boldsymbol{x}_{\rm o}) + \frac{1}{2}\,\gamma\,\delta^{ij}\,\dot{w}(\boldsymbol{x}_{\rm o}), \tag{25}$$

$$\mathcal{C}(t, \boldsymbol{x}) = \frac{1}{10}\,(\gamma-2)\, r_{\rm o}^2\, (\dot{a}_{\rm o}^i\, r_{\rm o}^i), \tag{26}$$

$$\mathcal{D}^{ij}(t) = \delta^{ij}\,\gamma\, w(\boldsymbol{x}_{\rm o}), \tag{27}$$

$$\mathcal{D}^{ijk}(t) = \frac{1}{2}\,\gamma\,\left(\delta^{ij} a_{\rm o}^k + \delta^{ik} a_{\rm o}^j - \delta^{jk} a_{\rm o}^i\right). \tag{28}$$

Besides that, the rotational matrix $\mathcal{R}^a_i$ from the transformation of the spatial coordinates (21) is related to $\mathcal{C}_a$ from (16) as

$$\begin{aligned} c^2\, \mathcal{R}^a{}_i\, \dot{\mathcal{R}}^a{}_j \; &= \; -(1+\gamma)\, \varepsilon_{ijk}\, \mathcal{R}^a{}_k\, \mathcal{C}_a \\ &\quad +(1+2\gamma)\, v_{\rm o}^{[i}\, w_{,j]}(\boldsymbol{x}_{\rm o}) - 2(1+\gamma)\, w^{[i,j]}(\boldsymbol{x}_{\rm o}) + v_{\rm o}^{[i}\, \mathcal{R}^a{}_{j]}\, \mathcal{Q}_a + \mathcal{O}(c^{-2}). \end{aligned} \tag{29}$$

The dynamically non-rotating version of the CoMRS characterizes by $\mathcal{C}_a = 0$ (i.e. no Coriolis forces in the equations of motion of test particles) and, as it follows from (29), has specific spatial rotation relative to the BCRS consisting of geodetic (de Sitter), Lense-Thirring and Thomas precessions. The GCRS of the IAU is defined to be kinematically non-rotating, that is, it has no rotation of spatial axes relative to the BCRS (i.e. $\mathcal{R}^a_i = \delta^a_i$ in this case). Although Coriolis forces appear in the equations of motion relative to a kinematically non-rotating reference system, this choice is especially advantageous for modeling of astronomical observations, since no additional orientation-related re-calculations (e.g., of planetary ephemeris data) are necessary. The same arguments apply to the CoMRS: the kinematically non-rotating CoMRS is the most convenient choice of the orientation of the local coordinates. For the kinematically non-rotating CoMRS $\mathcal{R}^a_i = \delta^a_i$ and $\mathcal{C}_a$ has some specific non-zero value defined by (29). Below we retain $\mathcal{R}^a_i$ in the formulas, but it should be chosen to be identity matrix δ^a_i.

Matching of the CoMRS and BCRS metric tensors allows one to derive also the equations of motion of the CoMRS origin as well. The BCRS acceleration of the CoMRS origin (that is, the acceleration of the observer's center of mass) reads

$$\begin{aligned} a_{\rm o}^i &= w_{,i}(\boldsymbol{x}_{\rm o}) + \Delta a_{\rm o}^i \\ &\quad + \frac{1}{c^2}\Big(2(1+\gamma)\,\dot{w}^i(\boldsymbol{x}_{\rm o}) + \Big(\gamma v_E^2 - 2(\gamma+\beta)\, w(\boldsymbol{x}_{\rm o})\Big)\, w_{,i}(\boldsymbol{x}_{\rm o}) - (2\gamma+1)\, v_{\rm o}^i\,\dot{w}(\boldsymbol{x}_{\rm o}) \\ &\qquad - 2(1+\gamma)\, v_{\rm o}^j\, w^j_{,i}(\boldsymbol{x}_{\rm o}) - v_{\rm o}^i\, v_{\rm o}^j\, w_{,j}(\boldsymbol{x}_{\rm o})\Big) + \mathcal{O}(c^{-4}), \end{aligned} \tag{30}$$

where

$$\Delta a^i_{\rm o} \;=\; -\mathcal{R}^a_j\, \mathcal{Q}_a \left(\delta^{ij} - \frac{1}{c^2}\left(v_E^2\, \delta^{ij} + (2+\gamma)\, w(\boldsymbol{x}_{\rm o})\, \delta^{ij} + \frac{1}{2} v^i_{\rm o}\, v^j_{\rm o}\right)\right) \tag{31}$$

is the BCRS coordinate acceleration of the CoMRS origin relative to the co-moving locally inertial reference system. Clearly, $\Delta a^i_{\rm o}$ is proportional to $\mathcal{Q}_a$ and comes just from the re-calculation of the CoMRS-defined $\mathcal{Q}_a$ into the BCRS (see, the discussion of Eq. (4.29) of [23]). If the acceleration $\Delta a^i_{\rm o}$ is neglected, Eq. (30) coincides with the equation of time-like geodetic in the metric (2). If the gravitational fields of all N bodies can be described only by their masses (no further multipole moments of the gravitational field in the corresponding local reference system of each body), Eq. (30) produces Eq. (3) of [7].

From (20) with (22) and (23) follows that the CoMRS coordinate time $\mathcal{T}$ at the CoMRS origin $\mathcal{X}^a = 0$ (this is equivalent to $r^i_{\rm o} = 0$) coincides with the proper time τ of the test observer.

V. COORDINATE-INDUCED TETRAD FOR THE ORIGIN OF THE COMRS

Let us construct a tetrad (e.g., [26]) co-moving with the observer. Let us first introduce four vectors $\mathrm{e}^{\mu}_{(\alpha)}$ attached to a point on the worldline of the observer. Here index α is the tetrad index which runs from 0 to 3 and numerates the vectors. Index μ is a normal tensor index which can be lowered and raised by contracting with the metric tensor

$$\begin{aligned} \mathrm{e}_{(\alpha)\,\mu} \;&=\; g_{\mu\nu}\, \mathrm{e}^{\nu}_{(\alpha)}, \\ \mathrm{e}^{\mu}_{(\alpha)} \;&=\; g^{\mu\nu}\, \mathrm{e}_{(\alpha)\,\nu}. \end{aligned} \tag{32}$$

The four vectors are required to have the property

$$\mathrm{e}^{\mu}_{(\alpha)}\, \mathrm{e}_{(\beta)\,\mu} = \eta_{\alpha\beta}. \tag{33}$$

This equation implies that the vectors are orthogonal to each other, that vector $\mathrm{e}^{\mu}_{(0)}$ is unit and time-like, and that $\mathrm{e}^{\mu}_{(i)}$ are unit and space-like. Four additional vectors $\mathrm{e}^{(\alpha)\,\mu}$ (with tetrad index written as superscript) are then defined by

$$\mathrm{e}^{(\alpha)\,\mu}\, \mathrm{e}_{(\beta)\,\mu} = \delta^{\alpha}_{\beta}, \tag{34}$$

$\delta^{\alpha}_{\beta} = \mathrm{diag}(1,1,1,1)$ is the 4-dimensional Kronecker symbol. From (33) and (34) it is easy to show that the conversion between vectors $\mathrm{e}^{\mu}_{(\alpha)}$ and $\mathrm{e}^{(\alpha)\,\mu}$ can be performed simply by contracting with the Minkowski metric

$$\begin{aligned} \mathrm{e}^{\mu}_{(\alpha)} \;&=\; \eta_{\alpha\beta}\, \mathrm{e}^{(\beta)\,\mu}, \\ \mathrm{e}^{(\alpha)\,\mu} \;&=\; \eta^{\alpha\beta}\, \mathrm{e}^{\mu}_{(\beta)}, \end{aligned} \tag{35}$$

where $\eta^{\alpha\beta} = \mathrm{diag}(-1,1,1,1)$ is the matrix inverse to $\eta_{\alpha\beta}$. With the help of these vectors one can represent the metric tensor at the considered point of space-time as

$$g_{\mu\nu} = \mathrm{e}^{(\alpha)}_{\mu}\, \mathrm{e}_{(\alpha)\,\nu} = \eta_{\alpha\beta}\, \mathrm{e}^{(\alpha)}_{\mu}\, \mathrm{e}^{(\beta)}_{\nu}, \tag{36}$$

so that

$$ds^2 = \eta_{\alpha\beta}\, dx^{(\alpha)}\, dx^{(\beta)}, \tag{37}$$

with

$$dx^{(\alpha)} = \mathrm{e}^{(\alpha)}_{\mu}\, dx^{\mu}. \tag{38}$$

Eq. (37) shows that $dx^{(\alpha)}$ can be interpreted as observable infinitely small time intervals and distances in the infinitesimal neighborhood of the considered space-time point. For a tetrad co-moving with the observer the unit time-like vector $\mathrm{e}^{\mu}_{(0)}$ can be chosen to coincide with the 4-velocity of the observer, so that the projection $dx^{(0)} = \mathrm{e}^{(0)}_{\mu}\, dx^{\mu}$ coincides with the interval of the proper time of the observer $dx^{(0)} = d\tau$ between the events with coordinates x^{μ} and $x^{\mu} + dx^{\mu}$ both lying on the worldline of the observer. This means

$$\mathrm{e}^{(0)}_{\mu} = -g_{\mu\nu}\, \frac{dx^{\nu}_{\mathrm{o}}}{c\, d\tau}. \tag{39}$$

The vectors $\mathrm{e}^{(i)\,\mu}$ are then constrained by (36) up to arbitrary spatial rotation. This means that if $\mathrm{e}^{(i)}_{\mu}$ is a solution of (36) then

$$\bar{\mathrm{e}}^{(k)}_{\mu} = P^{k}_{j}\, \mathrm{e}^{(j)}_{\mu} \tag{40}$$

with arbitrary orthogonal matrix P^{k}_{j} is also a solution.

Considering the BCRS metric tensor (2) one gets from (39)

$$\begin{aligned}
\mathrm{e}^{(0)}_{0} &= 1 + \frac{1}{c^2}\left(\frac{1}{2}\, v_{\mathrm{o}}^2 - w(\boldsymbol{x}_{\mathrm{o}})\right) \\
&\quad + \frac{1}{c^4}\left(\frac{3}{8}\, v_{\mathrm{o}}^4 + \left(\gamma + \frac{1}{2}\right) v_{\mathrm{o}}^2\, w(\boldsymbol{x}_{\mathrm{o}}) + \left(\beta - \frac{1}{2}\right) w^2(\boldsymbol{x}_{\mathrm{o}})\right) + \mathcal{O}(c^{-5}), && (41)\\
\mathrm{e}^{(0)}_{i} &= -\frac{1}{c}\, v^{i}_{\mathrm{o}} + \frac{1}{c^3}\left(-\frac{1}{2}\, v_{\mathrm{o}}^2\, v^{i}_{\mathrm{o}} - (2\,\gamma + 1)\, w(\boldsymbol{x}_{\mathrm{o}})\, v^{i}_{\mathrm{o}} + 2\,(1 + \gamma)\, w^{i}(\boldsymbol{x}_{\mathrm{o}})\right) + \mathcal{O}(c^{-5}). && (42)
\end{aligned}$$

The following partial solution for $\mathrm{e}^{(a)}_{\mu}$ is then possible:

$$\begin{aligned}
\mathrm{e}^{(a)}_{0} &= -\frac{1}{c}\, v^{a}_{\mathrm{o}}\left(1 + \frac{1}{c^2}\left(\frac{1}{2}\, v_{\mathrm{o}}^2 + \gamma\, w(\boldsymbol{x}_{\mathrm{o}})\right)\right) + \mathcal{O}(c^{-5}), && (43)\\
\mathrm{e}^{(a)}_{i} &= \delta^{ai} + \frac{1}{c^2}\left(\frac{1}{2}\, v^{a}_{\mathrm{o}}\, v^{i}_{\mathrm{o}} + \gamma\, w(\boldsymbol{x}_{\mathrm{o}})\, \delta^{ai}\right) + \mathcal{O}(c^{-4}). && (44)
\end{aligned}$$

This solution for $\mathrm{e}^{(a)}_{i}$ shows that we have chosen the space-like vectors of the tetrad is such a way that they show no spatial rotation relative to the spatial axes of the BCRS. It is easy to see from (20)–(28) that this tetrad is the coordinate basis of the kinematically non-rotating CoMRS (i.e. of the CoMRS with $\mathcal{R}^{a}_{i} = \delta^{a}_{i}$) on the worldline of its origin

$$\mathrm{e}^{(\alpha)}_{\mu} = \left.\frac{\partial \mathcal{X}^{\alpha}}{\partial x^{\mu}}\right|_{\mathcal{X}^i=0}. \tag{45}$$

Therefore, the CoMRS implies also adopting a particular tetrad co-moving with the observer. Tetrad (41)–(44) is induced by the CoMRS coordinates at the origin of the CoMRS in the sense of Section 3.4.2 of [27]. This tetrad can be used to model certain kind of observables (e.g., proper directions as directions relative to the tetrad (41)–(44)). However, the CoMRS is more than just a tetrad and allows one to use all the power of the theory of local reference systems as mentioned in Section I.

If one adopts a dynamically non-rotating CoMRS with $\mathcal{C}_a \equiv 0$ and $\mathcal{R}^a_i$ defined by (29), the corresponding tetrad (45) will be Fermi-Walker transported, so that the Fermi rotation coefficients of that tetrad vanish. This is, however, an unnecessary complication for space astrometry, where tetrad (41)–(44) and the kinematically non-rotating CoMRS are more convenient.

The tetrad (41)–(44) is written for arbitrary velocity of the observer $v^i_{\rm o}$. The tetrads $\widetilde{\rm e}^{(\alpha)}_\mu$ and ${\rm e}^{(\alpha)}_\mu$ defined by (41)–(44) with two different velocities $\widetilde{v}^i_{\rm o}$ and $v^i_{\rm o}$, respectively, are related to each other by a Lorentz transformation Λ^α_β plus additional spatial rotation $\mathcal{P}^i_{\ j}$ of space-like vectors:

$$
\begin{aligned}
{\rm e}^{(0)}_\mu &= \Lambda^0_\alpha\,\widetilde{\rm e}^{(\alpha)}_\mu, && (46)\\
{\rm e}^{(i)}_\mu &= \mathcal{P}^i_{\ j}\,\Lambda^j_\alpha\,\widetilde{\rm e}^{(\alpha)}_\mu, && (47)
\end{aligned}
$$

where

$$
\begin{aligned}
\Lambda^0_0 &= \Gamma,\\
\Lambda^0_a &= -\frac{1}{c}\,\nu^a\,\Gamma,\\
\Lambda^i_0 &= -\frac{1}{c}\,\nu^i\,\Gamma,\\
\Lambda^i_a &= \delta^{ia} + \frac{1}{c^2}\,\frac{\Gamma^2}{1+\Gamma}\,\nu^i\,\nu^a,\\
\Gamma &= \left(1-\frac{1}{c^2}\,\nu^k\,\nu^k\right)^{-1/2}, && (48)
\end{aligned}
$$

and

$$
\mathcal{P}^i_{\ j} = \delta^{ij} + \frac{1}{2\,c^2}\,\left(\widetilde{v}^i_{\rm o}\,v^j_{\rm o} - \widetilde{v}^j_{\rm o}\,v^i_{\rm o}\right) + \mathcal{O}(c^{-4}). \qquad (49)
$$

Here, the parameter ν^i of the Lorentz transformation is the BCRS velocity $v^i_{\rm o}$ of the second observer as seen by the first observer having the BCRS velocity $\widetilde{v}^i_{\rm o}$ which can be calculated as

$$
\begin{aligned}
\nu^i &= c\,\frac{d\widetilde{x}^{(i)}}{d\widetilde{x}^{(0)}} = c\,\frac{\widetilde{\rm e}^{(i)}_\mu\,dx^\mu}{\widetilde{\rm e}^{(0)}_\mu\,dx^\mu} = c\,\frac{\widetilde{\rm e}^{(i)}_0 + \widetilde{\rm e}^{(i)}_j\,v^j_{\rm o}/c}{\widetilde{\rm e}^{(0)}_0 + \widetilde{\rm e}^{(0)}_k\,v^k_{\rm o}/c}\\
&= v^i_{\rm o} - \widetilde{v}^i_{\rm o} + \frac{1}{c^2}\,\Biggl(\left(v^j_{\rm o}\,\widetilde{v}^j_{\rm o}\right)\left(v^i_{\rm o}-\widetilde{v}^i_{\rm o}\right) - \frac{1}{2}\left(\widetilde{v}^j_{\rm o}\,\widetilde{v}^j_{\rm o}\right)v^i_{\rm o} + \frac{1}{2}\left(v^j_{\rm o}\,\widetilde{v}^j_{\rm o}\right)\widetilde{v}^i_{\rm o}\\
&\qquad + (1+\gamma)\,w(\boldsymbol{x}_{\rm o})(v^i_{\rm o}-\widetilde{v}^i_{\rm o})\Biggr) + \mathcal{O}(c^{-4}). && (50)
\end{aligned}
$$

In the limit of special relativity Eq. (50) coincides with the special-relativistic velocity composition law. On the other hand, if the BCRS velocity of the first observer vanishes ($\widetilde{v}^i_{\rm o}=0$), Eq. (50) reproduces Eq. (12) of [7].

The reason for the appearance of the additional spatial rotation $\mathcal{P}^i{}_j$ is the well-known fact that two subsequent Lorentz transformations without spatial rotation are equivalent to the one Lorentz transformation with spatial rotation. This additional spatial rotation and its consequences for kinematically non-rotating astronomical reference systems are discussed in [28, 29]. The matrix (49) is equal to identity matrix $\delta^i{}_j$ if $\widetilde{v}^i_{\rm o}=0$, so that if $\widetilde{v}^i_{\rm o}=0$ the transformation between $\widetilde{\rm e}^{(\alpha)}_\mu$ and ${\rm e}^{(\alpha)}_\mu$ is a pure Lorentz transformation with parameter $\nu^i=v^i_{\rm o}\,\left(1+c^{-2}\,(1+\gamma)\,w(\boldsymbol{x}_{\rm o})\right)+\mathcal{O}(c^{-4})$. Using the Lorentz transformation in its closed form allows one to get the expressions for the tetrad in the first post-Minkowskian approximation, that is, retaining terms of first order in G and of all orders in $|v^i_{\rm o}|/c$.

VI. OBSERVED DIRECTION OF THE LIGHT PROPAGATION

Let us compute explicitly the relation between the unit coordinate direction of light propagation n^i in the BCRS and the observed direction to the light source relative to the tetrad (41)–(44). Let $x^\mu_p(t)$ be the coordinates of the photon parametrized by the coordinate time t. It is clear that the observed direction should be defined with respect to the tetrad vectors ${\rm e}^{(\alpha)}_\mu$ as

$$s^{(a)}=-\frac{dx^{(a)}_p}{dx^{(0)}_p}=-\frac{{\rm e}^{(a)}_\mu\,dx^\mu_p}{{\rm e}^{(0)}_\mu\,dx^\mu_p}=-\frac{{\rm e}^{(a)}_0+{\rm e}^{(a)}_i\,p^i}{{\rm e}^{(0)}_0+{\rm e}^{(0)}_j\,p^j},\tag{51}$$

where $p^i=\frac{1}{c}\frac{dx^i_p}{dt}$ is the coordinate light velocity at the point of observation $x^i=x^i_{\rm o}$. The differentials of x^μ_p are calculated here along the light ray $x^\mu_p(t)$ at the point of observation. However, because of (45) the direction $s^{(a)}$ coincides with the CoMRS velocity of the light propagation at the origin of the CoMRS (that is with the velocity p^i transformed into the CoMRS using the coordinate transformation (20)–(21)):

$$s^a=-\frac{d\mathcal{X}^a_p}{d\mathcal{X}^0_p}=-\frac{\frac{\partial\mathcal{X}^a}{\partial x^\mu}\,dx^\mu_p}{\frac{\partial\mathcal{X}^0}{\partial x^\mu}\,dx^\mu_p}=-\frac{\frac{\partial\mathcal{X}^a}{\partial x^0}+\frac{\partial\mathcal{X}^a}{\partial x^i}\,p^i}{\frac{\partial\mathcal{X}^0}{\partial x^0}+\frac{\partial\mathcal{X}^0}{\partial x^j}\,p^j}=s^{(a)}.\tag{52}$$

Here $\mathcal{X}^\alpha_p(\mathcal{T})$ is the worldline $x^\mu_p(t)$ of the light ray expressed in the CoMRS coordinates. All partial derivatives $\frac{\partial\mathcal{X}^\alpha}{\partial x^\mu}$ as well as the differentials $d\mathcal{X}^\alpha_p$ are calculated at the point of observation. Let us also note that (51) and (52) have one more interpretation. Let us consider two 4-vectors a^μ and b^μ and an observer with 4-velocity $u^\mu=\frac{dx^\mu_{\rm o}}{c\,d\tau}$. Let us also assume that both vectors a^μ and b^μ are not equal to $A\,u^\mu$, A being a constant. It is well known (e.g., [25, 27]) that by projecting each of these vectors into the observer's rest space and calculating the normalized scalar product of the projected vectors with respect to the metric $g_{\mu\nu}$ one gets the cosine of the angle θ between these two vectors as measured by the observer:

$$\overline{a}^\mu\;=\;P_{\mu\nu}\,a^\nu,\tag{53}$$
$$\overline{b}^\mu\;=\;P_{\mu\nu}\,b^\nu,\tag{54}$$

$$\cos\theta \;=\; \frac{\overline{a}_\mu\,\overline{b}^\mu}{\left(\overline{a}_\alpha\,\overline{a}^\alpha\right)^{1/2}\,\left(\overline{b}_\beta\,\overline{b}^\beta\right)^{1/2}}, \tag{55}$$

where $P_{\mu\nu} = g_{\mu\nu} + u_\mu\,u_\nu$ is the projection operator into the satellite's rest space. Using (39) and (36) one can write $P_{\mu\nu} = g_{\mu\nu} + e^{(0)}_\mu\,e^{(0)}_\nu = \delta_{ab}\,e^{(a)}_\mu\,e^{(b)}_\nu$. Therefore, the cosine of the observed angle θ_a between the incoming light ray with the wave vector $k^\mu = \frac{dx^\mu_p}{c\,dt} = (1, p^i)$ and a space-like vector of the triad $\mathrm{e}^{(a)}_\mu$ can be calculated as

$$\cos\theta_a = \frac{\mathrm{e}^{(a)}_\mu\,k^\mu}{\mathrm{e}^{(0)}_\nu\,k^\nu}. \tag{56}$$

Note that $P^\alpha_{\ \beta}\,\mathrm{e}^{(a)\,\beta} = \mathrm{e}^{(a)\,\alpha}$ and $\mathrm{e}^{(a)\,\alpha}\,\mathrm{e}^{(b)}_\alpha = \delta^{ab}$ and, therefore, vectors $\mathrm{e}^{(a)}_\alpha$ already lie in the observer's rest space and are normalized to unity. In (56) we used also that according to (33) and (39) $u_\alpha\,\mathrm{e}^{(a)\,\alpha} = 0$ for any $a = 1, 2, 3$. This technique to compute the cosines of the observed angles has been used in a slightly different form e.g. in [5, 6]. It is, however, clear that this technique is equivalent to the two above-mentioned ways to derive $s^{(a)}$ and the components of $s^{(a)}$ can be easily related to $\cos\theta_a$. Indeed, one has

$$\cos\theta_a \;=\; \frac{\mathrm{e}^{(a)}_\mu\,k^\mu}{\mathrm{e}^{(0)}_\nu\,k^\nu} = \frac{\mathrm{e}^{(a)}_\mu\,dx^\mu_p}{\mathrm{e}^{(0)}_\nu\,dx^\nu_p} = -s^{(a)}. \tag{57}$$

The difference in the sign between $\cos\theta_a$ and $s^{(a)}$ reflects the fact that $s^{(a)}$ is the direction toward the source while $\cos\theta_a$ characterizes the opposite direction of light propagation.

Now one can substitute (41)–(44) into (51) or (52) and expand the denominator into powers of c^{-1} to get the explicit relation between $s^{(a)}$ and p^i. The absolute value of the coordinate light velocity can be calculated from the fact that the light follows a null geodetic which means that in a reference system with metric tensor $g_{\alpha\beta}$ vector p^i satisfies the equation

$$g_{\mu\nu}\,k^\mu\,k^\nu = g_{00} + 2g_{0i}\,p^i + g_{ij}\,p^i\,p^j = 0. \tag{58}$$

Substituting the BCRS metric (2) into (58) one gets

$$|\boldsymbol{p}| = 1 - \frac{1}{c^2}\,(1+\gamma)\,w(\boldsymbol{x}_\mathrm{o}) + \frac{1}{c^3}\,2\,(1+\gamma)\,n^i\,w^i(\boldsymbol{x}_\mathrm{o}) + \mathcal{O}(c^{-4}), \tag{59}$$

where $|\boldsymbol{p}| = (p^1\,p^1 + p^2\,p^2 + p^3\,p^3)^{1/2}$ is the Euclidean norm of p^i. Combining (51)–(52) with (41)–(44) and (59) one gets

$$\begin{aligned}
\boldsymbol{s} = -\boldsymbol{n} \;+\; & \frac{1}{c}\,\boldsymbol{n}\times(\dot{\boldsymbol{x}}_\mathrm{o}\times\boldsymbol{n}) \\
+\; & \frac{1}{c^2}\left\{(\boldsymbol{n}\cdot\dot{\boldsymbol{x}}_\mathrm{o})\,\boldsymbol{n}\times(\dot{\boldsymbol{x}}_\mathrm{o}\times\boldsymbol{n}) + \frac{1}{2}\,\dot{\boldsymbol{x}}_\mathrm{o}\times(\boldsymbol{n}\times\dot{\boldsymbol{x}}_\mathrm{o})\right\} \\
+\; & \frac{1}{c^3}\left\{\left((\boldsymbol{n}\cdot\dot{\boldsymbol{x}}_\mathrm{o})^2 + (1+\gamma)\,w(\boldsymbol{x}_\mathrm{o})\right)\,\boldsymbol{n}\times(\dot{\boldsymbol{x}}_\mathrm{o}\times\boldsymbol{n}) + \frac{1}{2}\,(\boldsymbol{n}\cdot\dot{\boldsymbol{x}}_\mathrm{o})\,\dot{\boldsymbol{x}}_\mathrm{o}\times(\boldsymbol{n}\times\dot{\boldsymbol{x}}_\mathrm{o})\right\} \\
+\; & \mathcal{O}(c^{-4}),
\end{aligned} \tag{60}$$

Here $\boldsymbol{s} = s^a$, $\boldsymbol{n} = n^i$, and for any a^i and b^i the Euclidean scalar and vector products are denoted as $\boldsymbol{a} \cdot \boldsymbol{b} = \delta_{ij}\, a^i\, b^j = a^i\, b^i$ and $(\boldsymbol{a} \times \boldsymbol{b})^i = \varepsilon_{ijk}\, a^j\, b^k$, respectively. Eq. (60) coincides with Eq. (7) of [7]. The discussion of the Lorentz transformations of the tetrads $\mathrm{e}_{\mu}^{(\alpha)}$ at the end of Section V allows one to conclude that Eq. (60) can be re-written as a closed-form Lorentz transformation (see Section 5 of [7] for further details).

VII. RELATIVISTIC MODELING OF THE ROTATIONAL MOTION OF THE SATELLITE IN THE COMRS

In principle, one can consider the rotational motion of the satellite relative to the CoMRS. To this end, the post-Newtonian equations of rotational motion of a satellite relative to the CoMRS are necessary. These equations have been derived in [20] in the framework of general relativity and then extended to the PPN formalism in [22, 23]. The final multipole-expanded form of the equations is given in Section IX.F of [23]. These are differential equations for the post-Newtonian spin (total angular momentum) $\mathcal{S}^a$ of the satellite:

$$\dot{\mathcal{S}}^a = \mathcal{L}^a, \tag{61}$$

where $\mathcal{L}^a$ is the post-Newtonian external torque, that can be computed from the mechanical properties of the satellite (inertial moments, etc.) and the ephemeris data of the Solar system bodies (their BCRS positions, velocities, etc).

For a satellite on a heliocentric orbit with the semi-major axes close to that of the Earth orbit, the largest relativistic effect in its rotational motion relative to the kinematically non-rotating CoMRS is clearly the geodetic precession which is of the order of $\sim 2''/\mathrm{cty} \approx 2\ \mu\mathrm{as/hr}$.

VIII. ATTITUDE DESCRIPTION OF THE GAIA SATELLITE

It is, however, clear that these dynamical equations of rotational motion (at least the tiny relativistic corrections) play no role in the accurate attitude determination of the satellite. As it is discussed, e.g., in [2] the attitude of the satellite will be determined together with the astrometric parameters of the sources from aposteriori processing of the observational data. The attitude parameters are the parameters of the rotational matrix $P^a_{\ b}$ relating the CoMRS spatial axes $\mathcal{X}^a$ to the spatial axes $\overline{\mathcal{X}}^a$ of the reference system in which the satellite's body is fixed (the latter reference system is called Scanning Reference System (SRS) in [9]):

$$\overline{\mathcal{X}}^a = P^a_{\ b}\, \mathcal{X}^b. \tag{62}$$

The directly observable quantities for the missions like GAIA (i.e. for scanning astrometric satellites) are the coordinates $\overline{s}^a$ of the sources in the SRS tagged with the corresponding time of observation in the satellite's proper time

$$\overline{s}^a = P^a_{\ b}\, s^b, \tag{63}$$

where s^a is the vector defined by (51)–(52). The observables $\overline{s}^a$ should be first transformed from the SRS into the CoMRS with the aid of matrix $P^a_{\ b}$ and then the relativistic model as described e.g. in [7] should be applied to get the catalog positions of the sources. The matrix $P^a_{\ b}$ is clearly time-dependent and should be determined from the same observations

(a rough estimate of the matrix is provided by an apriori dynamical modeling of the satellite rotation). The matrix $P^a_{\ b}$ can be parametrized with some Euler-like angles or in any other suitable way.

In principle, any orientation of the CoMRS spatial axes can be used to accomplish this data processing scheme and thus determine both the astrometric parameters of the sources in the BCRS and the orientation of the satellites's body relative to the CoMRS. However, the kinematically non-rotating CoMRS represents a natural choice of the orientation of the local coordinates. Indeed, in this case the difference between the CoMRS positions of the sources and the catalogue positions comes from a number of well-understood effects like proper motion, parallax, light deflection (all calculated in the BCRS) and aberration (calculated as discussed in Section VI above). One can also argue that Eq. (60) takes its simplest form for the kinematically non-rotating CoMRS: for any other spatial orientation of the CoMRS the formula relating the observed direction to the source $\boldsymbol{s}$ to the BCRS direction $\boldsymbol{n}$ differs from (60) by additional spatial rotation which exactly vanishes for the kinematically non-rotating CoMRS.

One could in principle construct the tetrad directly for the SRS

$$
\begin{aligned}
\overline{\mathrm{e}}^{(0)}_{\mu} &= \mathrm{e}^{(0)}_{\mu}, \\
\overline{\mathrm{e}}^{(a)}_{\mu} &= P^a_{\ b}\,\mathrm{e}^{(b)}_{\mu},
\end{aligned}
\tag{64}
$$

where $\mathrm{e}^{(\alpha)}_{\mu}$ is the tetrad of the CoMRS defined by (41)–(44). This kind of tetrads was discussed in [5, 6]. The tetrad $\overline{\mathrm{e}}^{(\alpha)}_{\mu}$ can be used to compute the observed light direction $\overline{s}^a$ in the same way as described in Section VI. This way is totally equivalent to using first the CoMRS tetrad to compute s^a and then converting s^a into $\overline{s}^a$ by using (63). It is clear from (64) and, e.g., (51) since the contraction is associative and in particular $P^a_{\ b}\,\mathrm{e}^{(b)}_{\mu}\,dx^\mu = (P^a_{\ b}\,\mathrm{e}^{(b)}_{\mu})\,dx^\mu = P^a_{\ b}\,(\mathrm{e}^{(b)}_{\mu}\,dx^\mu)$. In our opinion, however, keeping the matrix $P^a_{\ b}$ in single formula (63) allows one to separate and simplify the relativistic and attitude models.

IX. CONCLUSIONS

The IAU 2000 relativity framework [10–12] provides not only a reasonable barycentric reference system for the whole solar system (BCRS) and a physically adequate geocentric reference system for the Earth neighborhood (GCRS). The IAU framework provides also suitable tools to model any kind of astronomical observations. As it was explicitly demonstrated above the same technique as used to construct the GCRS can be applied to define a physically adequate proper reference system of a test observer called CoMRS above. That reference system can be used to model observation processes of any kind. The coordinate basis of the CoMRS at its origin coincides with a particular form of tetrad comoving with the observer. This means that the CoMRS description of observables coincides with the classical tetrad representation in cases where a tetrad is sufficient for modeling. On the other hand, the CoMRS, being a well-defined 4-dimensional reference system, is much more that just a tetrad and can be used to model physical phenomena where spatial extension of the observer plays a role (e.g., its rotational motion). In principle, any spatial orientation of the CoMRS is possible. However, to model astronomical observations it is preferable to adopt the kinematically non-rotating CoMRS having no spatial rotation relative to the BCRS. This choice implies simplest possible models for observables. The attitude of the observer

can be then described by a 3-dimensional spatial rotation in the CoMRS. This allows one to separate the relativistic model and the attitude model and to simply both of them as much as possible.

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